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AN APPROACH TO THE SOLUTION OF DYNAMIC PROBLEMS FOR LAMINATED ELECTROELASTIC AND ANISOTROPIC MEDIA†

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The method of constructing the influence function for a multilayer isotropic medium (Green's matrix-symbol) described in [1] is extended to the case when the layers have electroelastic or anisotropic properties. A solution is obtained for piezoelectric crystals of class *6mm* of the hexagonal system (polarized along the z axis of the piezoelectric ceramics). The proposed approach is also applied to other problems of electroelasticity. When the piezoelectric and dielectric constants are equal to zero, Green's matrix-symbol is obtained for a transversely isotropic laminated medium. The dynamic contact problem is investigated and a numerical analysis of the effect of the anisotropic properties of the layers on the behaviour of a massive punch in the case of unsteady loading is carried out.

1. HARMONIC OSCILLATIONS OF AN ELECTROELASTIC LAYER

Suppose an electroelastic layer occupies the region $|z| \leq h, -\infty < x, y < \infty$. We will consider as the electroelastic material piezoelectric crystals polarized in the direction of the z axis). The mechanical loads $\mathbf{t}e^{-i\omega t}$, $\mathbf{r}e^{-i\omega t}$ and the electric-charge distribution density on the surface (the normal components of the electric induction or electric displacement) $d_1e^{i\omega t}$ and $d_2e^{i\omega t}$, respectively, are specified on the upper and lower boundaries of the layer.

Harmonic electroelastic oscillations of the layer are described by the following equations in the quasistatic approximation [2] (in dimensionless amplitudes of the parameters, the factor $e^{i\omega t}$ common for all the characteristics will be omitted everywhere)

$$\begin{gathered}\partial_1\theta+\Delta u+\Omega^2 u=0,\quad \partial_2\theta+\Delta\upsilon+\Omega^2\upsilon=0\\ \partial_3\theta^0+\Delta^0(w+e_1\psi)+\Omega^2 w=0\\ \partial_3\theta^*+\Delta^0(e_1 w-\varepsilon_1\psi)=0\end{gathered}\tag{1.1}$$

Here

$$\theta=\alpha_2 f+\partial_3(\alpha_3 w+e_0\psi),\quad \Delta=\alpha_1\Delta^0+\partial_3^2$$

$$\theta^0=\alpha_3 f+\partial_3(\delta_0 w+e_3\psi),\quad \Delta^0=\partial_1^2+\partial_2^2$$

$$\theta^*=e_0 f+\partial_3(e_3 w-\varepsilon_3\psi),\quad f=\partial_1 u+\partial_2\upsilon,\quad \alpha_2=\alpha_0-\alpha_1$$

$$\alpha_0=\frac{c_{11}}{c_{44}},\quad \beta_0=\frac{c_{12}}{c_{44}},\quad \gamma_0=\frac{c_{13}}{c_{44}},\quad \delta_0=\frac{c_{33}}{c_{44}},\quad \alpha_{1,2}=\frac{\alpha_0\mp\beta_0}{2}$$

$$\alpha_3=1+\gamma_0,\quad e_1=\frac{e_{15}l}{c_{44}},\quad e_2=\frac{c_{31}l}{c_{44}},\quad e_3=\frac{c_{33}l}{c_{44}}$$

$$e_0=e_1+e_2,\quad \varepsilon_1=\frac{\varepsilon_{11}l^2}{c_{44}},\quad \varepsilon_3=\frac{\varepsilon_{33}l^2}{c_{44}},\quad \Omega^2=\frac{\rho\omega^2a^2}{c_{44}}$$

The symbols ∂_i (i = 1, 2, 3) denote differentiation with respect to x/y, y/a and z/a, respectively, and x, y, z is a Cartesian system of coordinates. The components of the vector function $\mathbf{w}(x, y, z) = \{u, \upsilon,$

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w, $\psi\}$ are the horizontal and vertical displacements of points of the medium u, υ, w and the electric-field potential ψ; ω is the angular frequency of the oscillations, ρ is the density of the material, a is a characteristic linear dimension (for example, the half-width of the electrode), l is a normalizing factor which has the dimensions of electric field E, and c_{ij}, c_{ij}^E, e_{ij}, ε_{ij} are the dimensional elastic, piezoelectric and dielectric constants, respectively.

We will split the problem into two: symmetric and skew-symmetric.

The boundary conditions of the symmetric problem have the form

$$\begin{aligned} z=\pm h:\quad & \partial_3 u+\partial_1(w+e_1\psi)=\pm(t_1-r_1)/2 \\ & \partial_3\upsilon+\partial_2(w+e_1\psi)=\pm(t_2-r_2)/2 \\ & \gamma_0 f+\partial_3(\delta_0 w+e_3\psi)=(t_3+r_3)/2 \\ & e_2 f+\partial_3(e_3 w-\varepsilon_3\psi)=(d_1+d_2)/2 \end{aligned} \tag{1.2}$$

where $\mathbf{t}=\{t_1, t_2, t_3, d_1\}$, $\mathbf{r}=\{r_1, r_2, r_3, d_2\}$, $t_{1,2}$; $r_{1,2}$ are the horizontal components, and t_3 and r_3 are the vertical components of the vectors of the mechanical load, and $d_{1,2}$ are the normal components of the vectors of the electric induction, which acts on the upper and lower faces of the layer, respectively.

The boundary conditions of the skew-symmetric problem differ from (1.2) in the fact that $\pm(t_{1,2}-r_{1,2})$ is replaced by $t_{1,2}+r_{1,2}$, and t_3+r_3 by $\pm(t_3-r_3)$ while d_1+d_2 is replaced by $\pm(d_1-d_2)$.

We will seek the solution of system (1.1) in the form

$$u=\partial_1 f_1+\partial_2 f_2,\quad \upsilon=\partial_2 f_1-\partial_1 f_2,\quad w=w,\quad \psi=\psi \tag{1.3}$$

After substituting (1.3) into (1.1) we obtain after reduction in Fourier transforms with respect to x and y

$$\begin{aligned} & (-\lambda^2\alpha_0+\partial_3^2+\Omega^2)A-\lambda^2\partial_3(\alpha_3 W+e_0\Psi)=0 \\ & \alpha_3\partial_3 A+\partial_3^2(\delta_0 W+e_3\Psi)-\lambda^2(W+e_1\Psi)+\Omega^2 W=0 \\ & e_0\partial_3 A+\partial_3^2(e_3 W-\varepsilon_3\Psi)-\lambda^2(e_1 W-\varepsilon_1\Psi)=0 \\ & (-\lambda^2\alpha_1+\partial_3^2+\Omega^2)B=0\quad (\lambda^2=\alpha^2+\beta^2,\quad A=-\lambda^2 f_1,\quad B=-\lambda^2 f_2) \end{aligned} \tag{1.4}$$

where α and β are the parameters of the Fourier transform.

From the last equation of (1.4) we obtain

$$k^2-\alpha_1\lambda^2+\Omega^2=0,\quad k_4=\pm(\alpha_1\lambda^2-\Omega^2)^{1/2}=\pm\sigma_4$$

We will set up the characteristic equation of system (1.4) in the form

$$\begin{aligned} & \mu_1 k^6+\mu_2 k^4+\mu_3 k^2+\mu_4=0 \\ & \mu_1=-\varepsilon_3\delta_0-e_3^2,\quad \mu_2=\Omega^2(\mu_1-\varepsilon_3)+\lambda^2 c_1 \\ & \mu_3=-\varepsilon_3\Omega^4+\lambda^2\Omega^2(\varepsilon_3+\alpha_0\varepsilon_3+\chi_5+\chi_6)+\lambda^4 c_2 \\ & \mu_4=\Omega^4\lambda^2\varepsilon_1-\Omega^2\lambda^4(\chi_7+\varepsilon_1\alpha_0)+\lambda^6\alpha_0\chi_7 \\ & c_1=\varepsilon_3+e_0\chi_1-\alpha_3\chi_2+\chi_5-\alpha_0\mu_1 \\ & c_2=\alpha_3\chi_4-e_0\chi_3-e_1^2-\alpha_0\varepsilon_3-\alpha_0\chi_5-\varepsilon_1 \\ & \chi_1=e_0\delta_0-\alpha_3 e_3,\quad \chi_2=e_0 e_3+\alpha_3\varepsilon_3 \\ & \chi_3=e_0-\alpha_3 e_1,\quad \chi_4=e_0 e_1+\alpha_3\varepsilon_1 \\ & \chi_5=\varepsilon_1\delta_0+2e_1 e_3,\quad \chi_6=\varepsilon_1+e_0^2,\quad \chi_7=\varepsilon_1+e_1^2 \end{aligned} \tag{1.5}$$

As an analysis shows [2], this bicubic equation in the parameter k for known piezoelectric ceramics has two real roots $k_1=\pm\sigma_1$ and four complex-conjugate roots $\pm(k_2\pm ik_3)=\sigma_{2,3}$.

For the symmetric problem the solution can be represented in the form

$$A=\sum_{i=1}^{3} p_iW_i\,\mathrm{ch}\,\sigma_i z,\quad W=\sum_{i=1}^{3} W_i\,\mathrm{sh}\,\sigma_i z,\quad \Psi=\sum_{i=1}^{3} s_iW_i\,\mathrm{sh}\,\sigma_i z \tag{1.6}$$

$$B=b\,\mathrm{ch}\,\sigma_4 z$$

$$\begin{aligned} s_i&=(\chi_1\sigma_i^2+\Omega^2e_0-\chi_3\lambda^2)/(\chi_4\lambda^2-\chi_2\sigma_i^2)\\ p_i&=\lambda^2\sigma_i(\alpha_3+e_0s_i)/(\sigma_i^2-\alpha_0\lambda^2+\Omega^2)\end{aligned} \tag{1.7}$$

while for the skew-symmetric problem it can be represented in a similar form with the replacement $\mathrm{sh}(\cdot)\leftrightarrow\mathrm{ch}(\cdot)$.

The boundary conditions of the symmetric problem (1.2) can be converted to the form

$$\begin{aligned} z=\pm h{:}\quad &\partial_3A-\lambda^2(w+e_1\psi)=\mp\lambda^2Q_1^+/2\\ &\gamma_0A+\partial_3(\delta_0w+e_3\psi)=Q_3^+/2\\ &e_2A+\partial_3(e_3w-\varepsilon_3\psi)=Q_4^+/2\\ &\partial_3B=\mp\lambda^2Q_2^+/2\end{aligned} \tag{1.8}$$

For the skew-symmetric problem the quantities $\mp Q_1^+$, $\mp Q_2^+$ are replaced by $-Q_1^-$, $-Q_2^-$, while Q_3^+, Q_4^+ are replaced by $\pm Q_3^-$, $\pm Q_4^-$, respectively.

Here

$$\begin{aligned} Q_1^\pm&=i\lambda^{-2}[(T_1\mp R_1)\alpha+(T_2\mp R_2)\beta]\\ Q_2^\pm&=i\lambda^{-2}[(T_1\mp R_1)\beta+(T_2\mp R_2)\alpha]\\ Q_3^\pm&=T_3\pm R_3,\quad Q_4^\pm=D_1\pm D_2\end{aligned}$$

where T_i, R_i and D_i are the Fourier transforms of t_i, r_i and d_i, respectively.

From the last boundary condition of (1.8) we find b, and hence

$$B=-\lambda^2\,\mathrm{ch}\,\sigma_4z(2\sigma_4\,\mathrm{sh}\,\sigma_4h)^{-1}Q_2^+$$

The remaining boundary conditions of (1.8) give a system for determining the unknown coefficients W_i

$$\mathbf{LW}=\mathbf{F},\quad \mathbf{F}=\tfrac{1}{2}\{-\lambda^2Q_1^+,\ Q_3^+,\ Q_4^+\}$$

The elements of the matrix $\mathbf{L}=\|L_{ij}\|$ $(i,j=1,2,3)$ are as follows:

$$\begin{aligned} L_{1j}&=l_{1j}\,\mathrm{sh}\,\sigma_jh,\quad L_{ij}=l_{ij}\,\mathrm{ch}\,\sigma_jh,\quad i=2,3\\ l_{1j}&=-\lambda^2(1+e_1s_j)+\sigma_jp_j\\ l_{2j}&=\sigma_j(\delta_0+e_3s_j)+\gamma_0p_j\\ l_{3j}&=\sigma_j(e_3-\varepsilon_3s_j)+e_2p_j\end{aligned} \tag{1.9}$$

Taking into account the fact that $\lambda^2U=i\alpha A+i\beta B$ and $\lambda^2V=i\beta A+i\alpha B$, the solution of the symmetric problem (1.6) can be written in the form

$$\mathbf{W}(z)=\mathbf{A}_h^+(z)\mathbf{Q}^+,\quad \mathbf{Q}^+=\{Q_1^+,\ Q_2^+,\ Q_3^+,\ Q_4^+\},\quad \mathbf{W}=\{U,V,W,\Psi\}$$

We can similarly construct a solution of the skew-symmetric problem

$$\mathbf{W}(z)=\mathbf{A}_h^-(z)\mathbf{Q}^-$$

The matrices $A_h^\pm(z)\mathbf{Q}^-$ have the following structure

$$\mathbf{A}_h^{\pm}(z)=\left\|\begin{matrix}-i\alpha\lambda^2M_1^{\pm} & -i\beta\lambda^2N^{\pm} & i\alpha M_2^{\pm} & i\alpha M_3^{\pm}\\ -i\beta\lambda^2M_1^{\pm} & +i\alpha\lambda^2N^{\pm} & i\beta M_2^{\pm} & i\beta M_3^{\pm}\\ -\lambda^2K_1^{\pm} & 0 & K_2^{\pm} & K_3^{\pm}\\ -\lambda^2R_1^{\pm} & 0 & R_2^{\pm} & R_3^{\pm}\end{matrix}\right\|$$

$$K_i^+=\frac{1}{2\Delta^+}\sum_{j=1}^{3}m_{ij}s_j(z),\quad R_i^+=\frac{1}{2\Delta^+}\sum_{j=1}^{3}s_jm_{ij}s_j(z),\quad M_i^+=\frac{1}{2\lambda^2\Delta^+}\sum_{j=1}^{3}p_jm_{ij}c_j(z),$$

$$N^+=\operatorname{ch}\sigma_4z(2\sigma_4\lambda^2\operatorname{sh}\sigma_4h)^{-1},\quad \Delta^+=\sum_1^3 l_{1j}m_{1j}t_j$$

$$m_{11}=l_{22}l_{33}-l_{32}l_{23},\quad m_{12}=l_{31}l_{23}-l_{21}l_{33},\quad m_{13}=l_{21}l_{32}-l_{22}l_{31}$$

$$m_{21}=l_{13}l_{32}t_3-l_{12}l_{33}t_2,\quad m_{22}=l_{11}l_{33}t_1-l_{13}l_{31}t_3$$

$$m_{23}=l_{12}l_{31}t_2-l_{11}l_{32}t_1,\quad m_{31}=l_{12}l_{23}t_2-l_{13}l_{22}t_3$$

$$m_{32}=l_{13}l_{21}t_3-l_{11}l_{23}t_1,\quad m_{33}=l_{11}l_{22}t_1-l_{12}l_{21}t_2$$

$$s_j(z)=\frac{\operatorname{sh}\sigma_jz}{\operatorname{ch}\sigma_jh},\quad c_j(z)=\frac{\operatorname{ch}\sigma_jz}{\operatorname{ch}\sigma_jh},\quad t_j=\operatorname{th}\sigma_jh$$

The elements $\Delta^-, K_i^-, R_i^-, M_i^-, N^-$ are obtained by making the replacement sh $\leftrightarrow$ ch and $\Delta^+, K_i^+, R_i^+, M_i^+$, N^+, respectively.

The general solution of the problem is

$$\mathbf{W}(z)=\mathbf{A}_h^+(z)\mathbf{Q}^+ +\mathbf{A}_h^-(z)\mathbf{Q}^-$$

We will introduce two matrices of special form

$$\mathbf{C}^{\pm}=\left\|\begin{matrix}\pm i\alpha\lambda^{-2} & \pm i\beta\lambda^{-2} & 0 & 0\\ \pm i\beta\lambda^{-2} & \mp i\alpha\lambda^{-2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{matrix}\right\|$$

Then $\mathbf{Q}^{\pm} = \mathbf{C}^+\mathbf{T} \pm \mathbf{C}^-\mathbf{R}$, and the solution for the layer can be written as

$$\mathbf{W}(z)=\mathbf{B}_+(z)\mathbf{T}+\mathbf{B}_-(z)\mathbf{R} \tag{1.10}$$

$$\mathbf{B}_{\pm}(z)=(\mathbf{A}_h^+(z)\pm\mathbf{A}_h^-(z))\mathbf{C}^{\pm}=$$

$$=\left\|\begin{matrix}\alpha^2m_1^{\pm}+\beta^2n^{\pm} & \alpha\beta(m_1^{\pm}-n^{\pm}) & \pm i\alpha m_2^{\pm} & \pm i\alpha m_3^{\pm}\\ \alpha\beta(m_1^{\pm}-n^{\pm}) & \beta^2m_1^{\pm}+\alpha^2n^{\pm} & \pm i\beta m_2^{\pm} & \pm i\beta m_3^{\pm}\\ -i\alpha k_1^{\pm} & -i\beta k_1^{\pm} & \pm k_2^{\pm} & \pm k_3^{\pm}\\ -i\alpha r_1^{\pm} & -i\beta r_1^{\pm} & \pm r_2^{\pm} & \pm r_3^{\pm}\end{matrix}\right\| \tag{1.11}$$

$$m_i^{\pm}=M_i^-\pm M_i^+,\quad n^{\pm}=N^-\pm N^+,\quad k_i^{\pm}=K_i^-\pm K_i^+,\quad r_i^{\pm}=R_i^-\pm R_i^+$$

2. CONSTRUCTION OF GREEN'S MATRIX-SYMBOL FOR A MULTILAYER MEDIUM

Suppose the medium is a packet of N rigidly connected layers of thickness $H = 2(h_1 + \ldots + h_N)$ with rigidly clamped lower surface, and suppose the medium occupies the region $-H \leqslant z \leqslant 0, -\infty < x, y < \infty$. We will use the solution (1.10), (1.11) obtained above for the layer. We introduce a local system of coordinates for each layer

$$z_k=z+2(h_1+\ldots+h_{k-1})+h_k,\quad k=1,2,\ldots,N$$

We make a formal separation of the layers. Then the displacement of points of the kth layer, U, V

and W, and the electric potential Ψ will be given in dimensional parameters by the expression

$$\mathbf{W}_k(z_k)=[\mathbf{B}_+(z_k)\mathbf{T}_{k-1}+\mathbf{B}_-(z_k)\mathbf{T}_k]a/c_{44}^k,\quad k=1,2,\dots,N$$
$$\mathbf{W}_k=\{U,V,W,\Psi/l\},\quad \mathbf{T}_k=\{T_1,T_2,T_3,Dl\}$$

where T_k is a vector whose components are the forces and electric induction characterizing the interaction between the layers, and T_0 is a vector specified on the surface of the medium.

Note that to calculate the elements of the matrix $\mathbf{B}_\pm(z_k)$ in (1.10) and (1.11) one must use the elastic, piezoelectric and dielectric moduli of the corresponding layer.

We will write the matching conditions between the layers.

$$\mathbf{W}_k(-h_k)=\mathbf{W}_{k+1}(h_{k+1}),\quad k=1,2,\dots,N-1 \tag{2.1}$$

and the condition at the lower surface of the packet of layers

$$\mathbf{W}_N(-h_N)=0 \tag{2.2}$$

From (2.1) we have the recurrence relation

$$\begin{aligned}&\mathbf{B}_+(-h_k)\mathbf{T}_{k-1}+[\mathbf{B}_-(-h_k)-g_k\mathbf{B}_+(h_{k+1})]\mathbf{T}_k=\\&=g_k\mathbf{B}_-(h_{k+1})\mathbf{T}_{k+1},\quad g_k=c_{44}^k/c_{44}^{k+1}\end{aligned} \tag{2.3}$$

From (2.2) we determine

$$\mathbf{T}_N=-\mathbf{B}_-^{-1}(-h_N)\mathbf{B}_+(-h_N)\mathbf{T}_{N-1} \tag{2.4}$$

Using (2.3) and (2.4) we can express T_k in terms of the surface load T_0

$$\mathbf{T}_k=(-1)^k\textstyle\prod_{i=k}^1\mathbf{F}_i^{-1}\mathbf{B}_+(-h_i)\mathbf{T}_0,\quad k=1,2,\dots,N$$
$$\mathbf{F}_N=\mathbf{B}_-(-h_N),\quad \mathbf{F}_k=\mathbf{B}_-(-h_k)-g_k\mathbf{B}_+(h_{k+1})+$$
$$+g_k\mathbf{B}_-(h_{k+1})\mathbf{F}_{k+1}^{-1}\mathbf{B}_+(-h_{k+1}),\quad k=1,2,\dots,N-1$$

As a result, the displacements of the points of the multilayer medium and the electric potential will be determined in dimensionless form by the expression

$$\mathbf{W}(z)=\mathbf{K}(\alpha,\beta,z,\omega)\mathbf{T}_0 \tag{2.5}$$

$$\begin{aligned}&\mathbf{K}(\alpha,\beta,z,\omega)=(-1)^{k-1}(\mathbf{B}_+(z_k)-\mathbf{B}_-(z_k)\mathbf{F}_k^{-1}\mathbf{B}_+(-h_k))\times\\&\times\textstyle\prod_{i=k-1}^1\mathbf{F}_k^{-1}\mathbf{B}_+(-h_i)/g_k^0,\quad g_k^0=c_{44}^k/c_{44}^1,\quad k=1,2,\dots,N\end{aligned} \tag{2.6}$$

It has been established that as $\lambda\to\infty$, the asymptotic behaviour of the matrix K on the surface of the medium when $z=0$ is identical with the similar behaviour of K for the layer. We have $\mathbf{K}(\alpha,\beta,0,\omega)\sim\mathbf{B}_+(h_1)$ and

$$m_1^+\sim|\lambda|^{-3}M_1^0,\quad m_{2,3}^+\sim\lambda^{-2}M_{2,3}^0,\quad r_1^+\sim\lambda^{-2}R_1^0,\quad r_{2,3}^+\sim|\lambda|^{-1}R_{2,3}^0$$

$$k_1^+\sim\lambda^{-2}K_1^0,\quad k_{2,3}^+\sim|\lambda|^{-1}K_{2,3}^0,\quad n^+\sim|\lambda|^{-3}\alpha_1^{-1/2}$$
$$K_i^0=\frac{1}{\Delta^0}\sum_{j=1}^3\kappa_{ij},\quad R_i^0=\frac{1}{\Delta^0}\sum_{j=1}^3 b_j\kappa_{ij},\quad M_i^0=\frac{1}{\Delta^0}\sum_{j=1}^3 a_j\kappa_{ij}$$
$$b_j=\frac{\chi_1\eta_j^2-\chi_3}{\chi_4-\chi_2\eta_j^2},\quad a_j=\frac{\eta_j(\alpha_3+e_0b_j)}{\eta_j^2-\alpha_0},\quad \Delta^0=\det\mathbf{L}^0$$

κ_{ij} are the cofactors of the elements of the matrix $\mathbf{L}^0=\|\zeta_{ij}\|$ $(i,j=1,2,3)$, where

$$\zeta_{1j}=\eta_j a_j-1-e_1 b_j,\quad \zeta_{2j}=\eta_j(\delta_0+e_3 b_j)+\gamma_0 a_j$$
$$\zeta_{3j}=e_3-\varepsilon_3 b_j+e_2 a_j$$

and η_i are the roots of the cubic equation

$$\mu_1 k^3+c_1 k^2+c_2 k+\alpha_0\chi_7=0$$

The solution for a multilayer medium rigidly clamped to an elastic half-space is obtained by letting the thickness of the lower layer tend to infinity. By changing the system of coordinates to the form $z^* = z_N - h_N$ in the lower layer and passing to the limit we obtain the matrix

$$\mathbf{F}_N=0,\quad \mathbf{F}_{N-1}=\mathbf{B}_-(-h_{N-1})-g_{N-1}\mathbf{B}_+^\infty(0)$$
$$\mathbf{F}_k=\mathbf{B}_-(-h_k)-g_k\mathbf{B}_+(h_{k+1})+g_k\mathbf{B}_-(h_{k+1})\mathbf{F}_{k+1}^{-1}\mathbf{B}_+(-h_{k+1}),\quad k=1,2,\dots,N-2$$
$$z=z_k-2\sum_{i=1}^{k}h_i+h_k,\quad k=1,2,\dots,N-1;\quad z=z^*-2\sum_{i=1}^{N-1}h_i,\quad k=N$$

The matrix $\mathbf{B}_-^\infty(z^*)\equiv 0$, $\mathbf{B}_+^\infty(z^*) = 2\mathbf{A}_\infty^+(z^*)\mathbf{C}^+$ has the elements

$$k_i^+=\frac{1}{\Delta^\infty}\sum_{j=1}^{k}m_{ij}\exp(\sigma_j^N z),\quad r_i^+=\frac{1}{\Delta^\infty}\sum_{j=1}^{3}s_j m_{ij}\exp(\sigma_j^N z)$$

$$m_i^+=\frac{1}{\lambda^2\Delta^\infty}\sum_{j=1}^{k}p_j m_{ij}\exp(\sigma_j^N z),\quad n^+=\frac{\exp(\sigma_4^N z)}{\sigma_4^N\lambda^2},\quad \Delta^\infty=\det\mathbf{L}^\infty$$

where m_{ij} are the cofactors of the elements of the matrix $\mathbf{L}^\infty = \|l_{ij}\|$ $(i, j = 1, 2, 3)$, l_{ij}, p_j, s_j, given by (1.9) and (1.7), and σ_j^N are the roots of the characteristic equation (1.5) for the half-space.

In particular, for the half-space we have the simple formula

$$\mathbf{W}(z)=\mathbf{B}_+^\infty(z)\mathbf{T}_0$$

For a layer rigidly clamped to the half-space we obtain the displacement in the layer

$$\mathbf{W}(z)=(\mathbf{B}_+(z+h_1)-\mathbf{B}_-(z+h_1)\mathbf{F}_1^{-1}\mathbf{B}_+(-h_1)\mathbf{t}_0$$

in the half-space

$$\mathbf{W}(z)=-\mathbf{B}_+^\infty(z+2h_1)\mathbf{F}_1^{-1}\mathbf{B}_+(-h_1)\mathbf{t}_0/g_2^0,\quad \mathbf{F}_1=\mathbf{B}_-(-h_1)-g_1\mathbf{B}_+^\infty(0)$$

Applying an inverse Fourier transformation to (2.5) we obtain the integral representation of the solution for a multilayer medium

$$\mathbf{w}(x,y,z)=\frac{e^{-i\omega t}}{4\pi^2}\int\int_{-\infty}^{\infty}\mathbf{W}(z)e^{-i(\alpha x+\beta y)}d\alpha d\beta \tag{2.7}$$

3. THE MIXED DYNAMIC PROBLEM

Suppose we are given mixed boundary conditions on the surface of the medium $z = 0$. In the region S we are given the displacements and the electric potential

$$\mathbf{w}(x,y,0,\omega)=\mathbf{u}^0(x,y,\omega),\ (x,y)\in S$$

Outside the region S the stresses are zero and there is no normal component of the electric induction

$$\mathbf{t}^0=0,\ (x,y)\notin S$$

Then, from (2.7) we obtain a system of fourth-order integral equations

$$\iint\limits_S \mathbf{k}(x-\xi, y-\zeta, \omega)\mathbf{t}^0(\xi,\zeta,\omega)d\xi d\zeta = \mathbf{u}^0(x,y,\omega) \tag{3.1}$$

$$\mathbf{k}(x,y,\omega) = \frac{1}{4\pi^2}\int\limits_{\sigma_1}\int\limits_{\sigma_2} \mathbf{K}(\alpha,\beta,0,\omega)e^{-i(\alpha x+\beta y)}d\alpha d\beta$$

in the unknown vectors $\mathbf{t}^0\{t_1, t_2, t_3, d\}$. The contours of integration σ_1 and σ_2 are chosen in accordance with the principle of limiting absorption [3].

Problems of this kind on the excitation and oscillations by an electrode in an electroelastic layer with a rigidly clamped lower surface were considered in [4, 5]. Here a single electrode, as the simplest electroelastic wave transducer was modelled by a strip punch. Formulae were obtained which enable the contact pressures and the electric induction to be found over the whole region of contact of the electrode with the medium, including the boundary, and also the elastic and electric characteristics outside this region. The relations obtained remain true for an electroelastic multilayer medium; only the form of the integrand matrix function of the kernel of the system of integral equations which participate in the solutions constructed is changed. In this case the matrix function K is given by (2.6).

4. CONSTRUCTION OF GREEN'S MATRIX FUNCTION FOR A TRANSVERSELY ISOTROPIC LAMINATED MEDIUM

The method of constructing the matrix K considered in Section 2 can be extended to multilayer anisotropic media. As an example we will consider a transversely isotropic medium, the equations of which can be obtained by setting the piezoelectric and dielectric constants equal to zero in (1.1) and (1.2). In this case the three-dimensional load vector $\mathbf{t}^0\{t_1, t_2, t_3\}$ is specified on the surface of the medium and the displacements of the points of the laminated medium $\mathbf{w}\{u, \upsilon, w\}$ are given by (2.5)–(2.7). The matrix K is a 3×3 matrix and is obtained by deleting the fourth row and column from (1.11). After simplification the governing functions have the form

$$\begin{aligned}
&M_1^+ = \frac{l_2c_1(z)-l_1c_2(z)}{2\lambda^2\Delta^+},\quad M_2^+ = \frac{b_1t_1c_2(z)-b_2t_2c_1(z)}{2\lambda^2\Delta^+}\\
&N^+ = \frac{\operatorname{ch}\sigma_4 z}{2\lambda^2\sigma_4\operatorname{sh}\sigma_4 h},\quad \sigma_4^2 = \alpha_1\lambda^2-\Omega^2\\
&K_1^+ = \frac{l_2a_1s_1(z)-l_1a_2s_2(z)}{2\Delta^+},\quad K_2^+ = \frac{b_1a_2t_1s_2(z)-b_2a_1t_2s_1(z)}{2\Delta^+}\\
&\Delta^+ = b_1l_2t_1-b_2l_1t_2,\quad a_i = \sigma_i\alpha_3(\lambda^2-\Omega^2-\delta_0\sigma_i^2)^{-1}\\
&l_i = \gamma_0+\delta_0a_i\sigma_i,\quad b_i = \sigma_i-\lambda^2a_i,\quad i=1,2
\end{aligned} \tag{4.1}$$

where σ_i are the roots of the biquadratic equation

$$\delta_0k^4-[\lambda^2(1+\delta_0\alpha_0-\alpha_3^2)-\Omega^2(\delta_0+1)]k^2+(\lambda^2-\Omega^2)(\alpha_0\lambda^2-\Omega^2)=0\quad k_{1,2}=\pm\sigma_{1,2} \tag{4.2}$$

In the special case of an isotropic medium we have

$$\alpha_0=\delta_0=\frac{\lambda+2\mu}{\mu}=2\frac{1-\nu}{1-2\nu},\quad \beta_0=\gamma_0=\frac{\lambda}{\mu}=2\frac{\nu}{1-2\nu}$$

$$\alpha_1=1,\quad \alpha_2=\alpha_3=\frac{\lambda+\mu}{\mu}=\frac{1}{1-2\nu}$$

where λ, μ are the Lamé constants and ν is Poisson's ratio.

5. THE UNSTEADY CONTACT PROBLEM

We will consider the dynamic contact problem of the interaction between a punch of mass m with a transversely isotropic laminated medium occupying the region $-\infty \leqslant x, y \leqslant \infty$, $-H \leqslant z \leqslant 0$. We will assume that the punch has a flat base S and that the centre of mass of the punch coincides with the origin of coordinates. A load which varies with time t in a specified way acts on the punch. The load, reduced to the centre of mass, is split into a force component $\mathbf{P}(t) = \{P_1, P_2, P_3\}$ and a moment $\mathbf{M}(t) = \{M_1, M_2, M_3\}$. The system is at rest at the initial instant of time. The displacements of the points of the punch $\mathbf{u}^0(t) = \{u_1^0, u_2^0, u_3^0\}$ are defined in the form $\mathbf{u}^0 = \mathbf{u} + \boldsymbol{\varphi} \times \mathbf{r}$ or

$$u_1^0 = u_1 - \varphi_3 y, \quad u_2^0 = u_2 + \varphi_3 x, \quad u_3^0 = u_3 + \varphi_1 y - \varphi_2 x$$

where u_1, u_2 and u_3 are the horizontal and vertical components of the displacement of the centre of mass of the punch and $\boldsymbol{\varphi}$ is the vector of the angles of rotation about the centre of mass of the punch.

The equations of motion of the punch in Laplace transforms take the form

$$mp^2 u_i = -P_i + \sum_{k=1}^{3} Q_i^k u_k + \sum_{k=4}^{6} Q_i^k \varphi_{k-3} \tag{5.1}$$

$$J_i p^2 \varphi_i = -M_i + \sum_{k=1}^{3} R_i^k u_k + \sum_{k=4}^{6} R_i^k \varphi_{k-3}, \quad i = 1,2,3$$

System (5.1) contains six unknown functions u_i, φ_i ($i = 1, 2, 3$) which also define the motion of the punch of mass m; J_i are the moments of inertia about the x, y and z axes. The corresponding forces and moments which occur in the region of contact between the body and the medium will be defined by the formulae

$$R_1^k = \iint_S q_3^k y \, dS, \quad R_2^k = -\iint_S q_3^k x \, dS$$

$$R_3^k = \iint_S (q_2^k x - q_1^k y) \, dS, \quad Q_i^k = \iint_S q_i^k \, dS, \quad k = 1,2,3$$

The vector functions $\mathbf{q}^k \{q_1^k, q_2^k, q_3^k\}$ ($k = 1, 2, \ldots, 6$) are the solutions of the system of integral equations (3.1) for the right-hand side, when only one of the components u_1, u_2, u_3, φ_1, φ_2, φ_3, is non-zero, in this case having unit value. In the unsteady problem we must replace the oscillation frequency ω by ip.

For a strip-shaped punch of width $2a$, Eqs (5.1) simplify and the displacements of the punch in Laplace transforms will be given by the expressions

$$u_1 = [P_1(Jp^2 + R^3) - MQ_1^3]\Delta_0^{-1}, \quad u_2 = P_2(mp^2 + Q_2^2)^{-1}$$

$$\varphi = [M(mp^2 + Q_1^1) - P_1 Q_1^3]\Delta_0^{-1} \tag{5.2}$$

$$\Delta_0 = (mp^2 + Q_1^1)(Jp^2 + R^3) - (Q_1^3)^2$$

$$R^3 = \int_{-a}^{a} q_2^3(x) x \, dx, \quad Q_i^k = \int_{-a}^{a} q_i^k(x) \, dx, \quad k = 1,2,3; \; i = 1,2$$

where p is the Laplace transform parameter, $\mathbf{q}^k \{q_1^k, q_2^k\}$ are the solutions of the corresponding dynamic contact problems

$$\mathbf{K}\mathbf{q}^1 = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix}, \quad \mathbf{K}\mathbf{q}^2 = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}, \quad \mathbf{K}\mathbf{q}^3 = \begin{Vmatrix} 0 \\ x \end{Vmatrix}$$

and $R^3(p)$, $Q_i^k(p)$ are constructed by the fictional absorption method in analytic form, which simplifies the inverse Laplace transformation of (5.2) [1, 6]. Here the numerical Laplace inversion is carried out using Filon's method which, in the problems considered, enables high calculation accuracy to be obtained.

Problems of the action of an unsteady load on a strip-shaped punch in contact with isotropic multilayer bases were investigated in detail in [1, 6]. The effect of the anisotropic properties of the layers on the displacement of the punch and the nature of the unsteady process in the medium can be investigated using the solutions constructed in [1, 6]. The governing functions $M_i^{\pm}, N^{\pm}, K_i^{\pm}, \Delta^{\pm}$ for the matrix K must be taken in the form (4.1), and one must take into account the different behaviour of the matrices K at infinity for isotropic and anisotropic media.

Figure 1 shows the vertical displacements of a punch in contact without friction with a transversely isotropic layer of thickness $H = 0.5$, rigidly clamped to an undeformable base. A vertical load $P(t) = H(t - 0.3)$ acts on the punch. Curve 1 illustrates the isotropic layer with parameters $\alpha_0 = \delta_0 = 3.5$, $\gamma_0 = 1.5$, which corresponds to $\nu = 0.3$. Curves 2 and 3 correspond to a transversely isotropic medium with $\delta_0 = 2.0$ and 5.0 ($\alpha_0 = 3.5$, $\gamma_0 = 1.5$). It was established that a change in the parameters α_0 and γ_0 has no effect on the amplitude and period of the oscillations of the punch and is due to the form of the functionals (5.3), defined when $\lambda = 0$ taking expressions (4.1) and (4.2), which participate in the solution, into account.

Figure 2 illustrates the displacements of a punch which interacts with a two-layer medium. The upper layer is an isotropic medium of thickness $2h_1 = 0.4$ and the lower layer is a transversely isotropic medium of thickness $2h_2 = 0.4$. In this case the anisotropy in the lower layer is introduced by changing the parameter $\delta_0 = \varepsilon\alpha_0$. A vertical load $P(t) = H(t - 0.1)$ acts on the punch. Curve 1 illustrates the problem for an isotropic layer of thickness $H = 2h_1 + 2h_2 = 0.8$ ($\alpha_0 = \delta_0 = 3.5$, $\gamma_0 = 1.5$). Curves 2 and 3 correspond to $\varepsilon = 0.5$ and 1.5 in the lower layer. The curves are identical up to the instant when the wave reflected from the interface of the layers arrives. The amplitude and period of the oscillations of the punch after removal of the load decreases as ε increases.

The calculations were carried out for viscoelastic media. In this case $\omega = ipe^{-\zeta}$, where ζ is the viscosity parameter of the medium, $0 \leq 2\zeta \leq 1$ (the elasticity constants are complex quantities of the form $c_{ij}e^{2i\zeta}$). The numerical inverse Laplace transformation is carried out along the real axis. We took $m = 1$ and $\zeta = 0.2$ in the calculations.

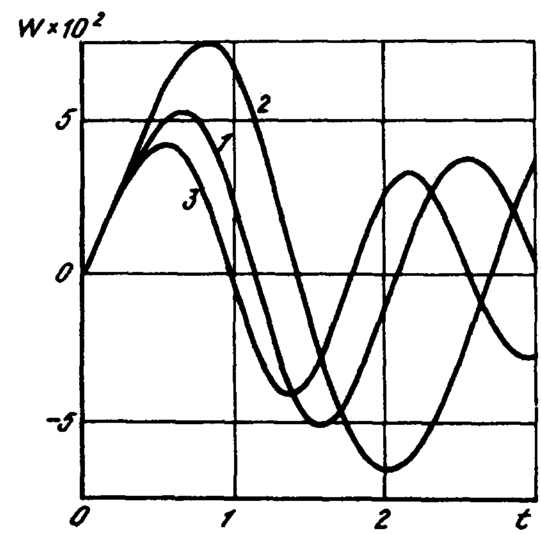

Fig. 1.

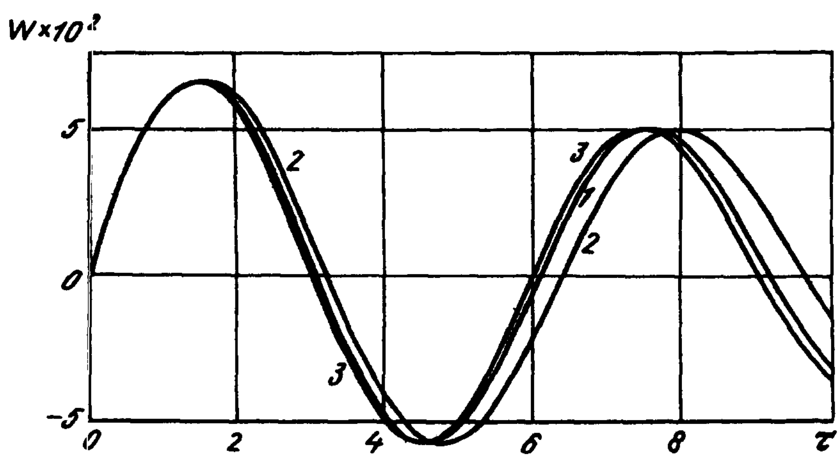

Fig. 2.

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